27.03.14

F8WS16

$$\boxed{\begin{aligned} dv &= r^2 \sin\vartheta \, dr\, d\vartheta\, d\phi \\ da &= r^2 \sin\vartheta \, d\vartheta\, d\phi \end{aligned}}$$

$$\boxed{dv = dx\,dy\,dz = \rho\, d\rho\, d\phi\, dz = r^2 \sin\vartheta\, dr\, d\vartheta\, d\phi}$$

$r\,d\vartheta$

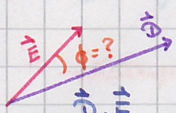
$\rho\, d\phi = r \sin\vartheta\, d\phi$

S24 **Bsp. 2.5**

$$\vec{E} = \frac{E}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{und } \overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1{,}5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \varepsilon_0 \left(1{,}5\, \hat{x}\hat{x} + 2\,\hat{y}\hat{y} + 3\,\hat{z}\hat{z}\right)$$

Flussdichte $\vec{D} = \overleftrightarrow{\varepsilon}\vec{E} = \varepsilon_0 \cdot \frac{E}{\sqrt{3}} \begin{bmatrix} 1{,}5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{\varepsilon_0 E}{\sqrt{3}} \cdot \begin{pmatrix} 1{,}5 \\ 2 \\ 3 \end{pmatrix} \Rightarrow |\vec{D}| = \|\vec{D}\| = D = \varepsilon_0 \cdot E \cdot \frac{1}{\sqrt{3}} \cdot \sqrt{1{,}5^2 + 2^2 + 3^2}$

$$= \sqrt{\frac{15{,}25}{3}}\, \varepsilon_0 \cdot E = \underline{2{,}2546\, \varepsilon_0 \cdot E}$$




$$\cos\phi = \frac{\vec{D}\cdot\vec{E}}{D \cdot E} = \frac{\frac{\varepsilon_0 \cdot E^2}{3} \cdot (1 \cdot 1{,}5 + 1 \cdot 2 + 1 \cdot 3)}{E \cdot \varepsilon_0 \cdot E \;\; 2{,}2546} \approx 0{,}9611 \Rightarrow \phi = \arccos(0{,}961) = 0{,}28025 \text{ rad} = \underline{16{,}05°}$$

S25 **2.4 Differentiation**

$$\boxed{f'(x) = \frac{df}{dx}} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Steigung der Tangente     Steigung der Sekante

$$\boxed{df = f'(x)\,dx} \quad \text{bzw.} \quad \underbrace{\Delta f}_{f(x) - f(x_0)} \approx \underbrace{f'(x)}_{f'(x_0)} \underbrace{\Delta x}_{(x - x_0)}$$

$$\Rightarrow f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$ → erstes Element der Taylorreihe

S27 **Partielle Ableitung**

$\frac{df}{dx}, \frac{df}{dy}, \frac{df}{\partial z}$

Bewegung in x-Richtung $df = \frac{\partial f}{\partial x} dx$

-"- y-Richtung $df = \frac{\partial f}{\partial y} dy$

-"- z-Richtung $df = \frac{\partial f}{\partial z} dz$

$$df = \frac{\partial f}{dx} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

beliebige Richtung     totale Differential

$$= \underbrace{\left(\frac{\partial f}{\partial x}\;\frac{\partial f}{\partial y}\;\frac{\partial f}{\partial z}\right)}_{\text{grad } f} \underbrace{\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}_{\vec{ds}}$$

$$\boxed{df = \text{grad } f \cdot \vec{ds} \quad \text{mit grad } f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}} = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

Änderung des Feldwertes $f(x,y,z)$ bei Bewegung um das kleine Stück $\vec{ds}$

$\vec{ds} = \hat{e}\,ds \Rightarrow df = \text{grad } f\, \hat{e}\, ds \Rightarrow \boxed{\frac{df}{ds} = \hat{e} \text{ grad } f}$ ≙ Ableitung / Steigung von f in die Richtung $\hat{e}$ ≙ Richtungsableitung

Bsp.: $\hat{e} = \hat{x} \Rightarrow \frac{df}{ds} = \hat{x} \cdot \text{grad } f = \frac{\partial f}{\partial x}$

In welche Richtung $\hat{e}$ wird $\frac{df}{ds}$ maximal? $\underset{\text{gegeben}}{\hat{a}\,\hat{b}} \to \max \Rightarrow \underset{\text{Richtungsparallel}}{\hat{a} \uparrow\uparrow \hat{b}} \to \frac{df}{ds} = \max$ falls $\hat{e} \uparrow\uparrow \text{grad } f$

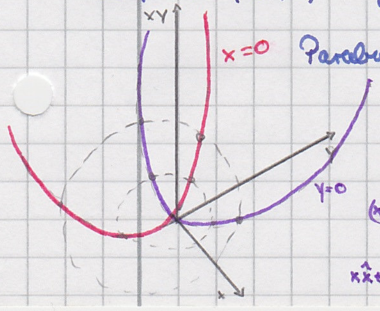
Richtung ↳ grad f ≙ Vektor in Richtung des steilsten Anstiegs von $f(x,y,z)$

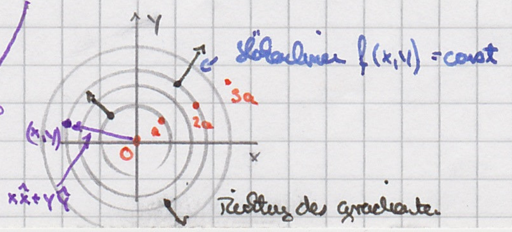
Länge / Betrag ↳ ‖grad f‖ ≙ Ableitung in dieser Richtung

Seite 30 **Bsp. 2.7** $f(x,y) = a(\underbrace{x^2 + y^2}_{\rho^2})$    $\text{grad } f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} = 2ax\hat{x} + 2ay\hat{y}$

$= 2a(x\hat{x} + y\hat{y})$ radial nach außen

Paraboloid (rotierende Parabel)



$f_{Bsp}(\rho,\phi) = a\rho^2 \Rightarrow grad(f_{Bsp}) = \hat{\rho}\frac{\partial f}{\partial \rho} + \underbrace{\hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi}}_{=0} = \hat{\rho}\cdot 2a\cdot\rho$ zeigt radial nach außen

**Schreibweisen** $\sigma\, da$, $\vec{D}\, d\vec{a}$, $\vec{j}\, d\vec{a}$, $\rho_m\, dv$

$\uparrow \Delta a \to 0,\ \Delta v \to 0$

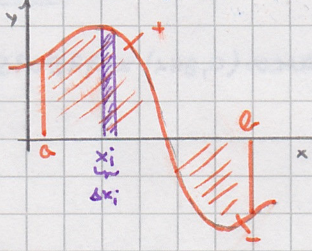
$\sigma\, \Delta a$, $\vec{D}\, \Delta\vec{a}$, $\vec{j}\, \Delta\vec{a}$, $\Delta m = \rho_m \Delta v$

S.32 **2.5 Integration** $I = j\cdot A$

$\oint \vec{D}\, d\vec{a}$ $\qquad \int E j\, dv$

$\int_a^b f(x)dx$ ; $\int_C \vec{E}\, d\vec{s}$ ; $I = \iint_A \vec{j}\, d\vec{a}$ ; $\oiint_{\partial V} \vec{D}\, d\vec{a}$ ; $\iiint_V \rho_m\, dv$ ; $\iiint_V E\cdot j\, dv$

**Klassische Integration** $\int_a^b f(x)dx \approx \sum_i f(x_i)\Delta x_i$




$$\boxed{\int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\Delta x_i}$$ mit $x_i \in \Delta x_i$ — Riemann-Interpretation

**Integral ≙ Summe über unendlich viele Teilbeiträge**

$$\boxed{\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)}$$ mit $f(x) = \frac{dF}{dx} = F'(x)$

Hauptsatz der Integralrechnung

**Integral** $\int_C f(\vec{r})\, \underset{\text{Wegelement}}{ds}$


$\rho_e = \frac{dm}{ds} ≙ \frac{\text{Masse}}{\text{Länge}}$

Masse $= \int_C \rho_e\, ds ≙ \sum \Delta m_c$

Länge der Raumkurve

$\sum \Delta s_i = \ell_c$

$$\boxed{\int_C f(\vec{r})ds = \lim_{\Delta s_i \to 0} \sum_i f(\vec{r}_i)\Delta s_i}$$

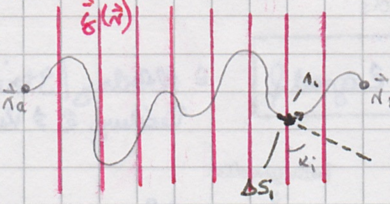
speziell $f(\vec{r}) = 1$ : $\int_C 1 ds = \int_C ds ≙ \Delta s_i = \ell_c$

S36 **Bsp 2.11** $\gamma = \frac{dQ}{dl} \to dQ = \gamma\, dl ≙ \gamma\, ds \to Q = \int_C \gamma\, ds = \int_0^{2\pi} \frac{\gamma_{max}}{2}(1+\cos\phi)\underbrace{R\cdot d\phi}$ Bogenlänge

$Q = \frac{\gamma_{max}}{2} 2\pi R = \pi\cdot R\, \gamma_{max}$

**Integral** $\int \vec{g}(\vec{r}) \underset{\text{Skalar}}{\cdot} d\vec{s}$ wofür?? $dW = \vec{F}\cdot d\vec{s} \Rightarrow \underbrace{\Delta W}_{\text{Gesamtänderung}} = \int \vec{F}\, d\vec{s}$ Kraft in Richtung des Weges

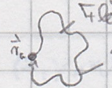
?




$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\, d\vec{s}$

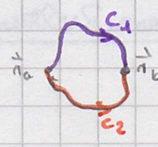
$$\boxed{\int_C \underbrace{\vec{g}(\vec{r})\cdot d\vec{s}}_{\vec{g}\cdot\vec{e}\, ds = g_e ds} = \lim_{\Delta s_i \to 0} \sum_i \underbrace{\vec{g}(\vec{r}_i)\cdot \Delta\vec{s}_i}_{g(\vec{r}_i)\Delta s_i \cos\alpha_i}}$$ = Linienintegral

**Spezialfall** $\vec{r}_a = \vec{r}_b$




Raumkurve $C = \partial A$ "$\partial A$" bedeutet "Rand von A" $\to$ $\boxed{\oint_{\partial A} \vec{g}\cdot d\vec{s}}$ ≙ Zirkulation (= Umlaufintegral) (wirbelfreie Felder)

$\oint_{\partial A \leftarrow \text{beliebig}} \vec{g}\cdot d\vec{s} = 0$ → Konservatives Feld




$\int_{C_1} \vec{g}\, d\vec{s} = \int_{C_2} \vec{g}\, d\vec{s}$

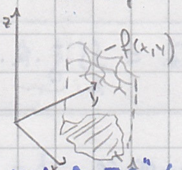
27.03.14

S38 FBW **Bsp. 2.12** $W_{12} = m \cdot g \cdot h = m \cdot g \cdot R$ => umgeführte Energie $\int_{C_{12}} (-\vec{F}) \vec{ds} = -\int_{C_{12}} \vec{F} \vec{ds}$

$$= -\int_{C_{12}} (-mg\hat{y}) \cdot \vec{ds} = \int_{C_{12}} mg\, \hat{y}\, \underbrace{\vec{ds}}_{R d\phi \hat{\phi}} = \int_0^{\pi/2} mg\, \hat{y}\hat{\phi}\, R d\phi = m \cdot g \int_0^{\pi/2} \underbrace{\hat{y}\hat{\phi}}_{1 \cdot 1 \cdot \cos\phi} d\phi = mgR [\sin\phi]_0^{\pi/2} = \underline{\underline{m \cdot g \cdot R}}$$

$ds = R \cdot d\phi$ ; Tabelle

S39 **Integral $\iint f(x,y) \underbrace{dy\, dx}_{da}$ oder $\iint f(\varrho, \phi) \underbrace{\varrho\, d\varrho\, d\phi}_{da}$** Bild Seite 39




$\vec{r}$ z.B. $(x,y)$

$$\boxed{\iint_A f(\vec{r})\, da = \lim_{\Delta a_i \to 0} \sum \underbrace{f(\vec{r}_i)\, \Delta a_i}_{\hat{=} \text{Volumen}}}$$ $\hat{=}$ Volumen zwischen x-y-Ebene und der Fläche $f(x,y)$ oberhalb der Fläche

← Flächenintegral über skalare Fkt. $f(\vec{r})$ über ebene Fläche $A$

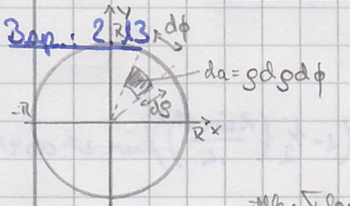
$$\boxed{\begin{array}{ll} da = dx\,dy & \Rightarrow \iint_A f(x,y)\, dx\, dy \\ da = \varrho\, d\varrho\, d\phi & \Rightarrow \iint_A f(\varrho,\phi)\, \varrho\, d\varrho\, d\phi \end{array}}$$

Kartesisch S20

Polar - S20 → $\boxed{\vec{da} = \varrho\, d\varrho\, d\phi\, \hat{z}}$ $\hat{=}$ Flächenelement auf Kreis (x-y-Ebene)

**Spezialfall:** $\boxed{f(\vec{r}) = 1: \quad \iint_A da = A}$ $\boxed{\vec{da} = r^2 \sin\vartheta\, d\vartheta\, d\phi\, \hat{r}}$ Flächenelement auf Kugeloberfläche

S41 **Bsp.: 2.13**




$$A = \iint da = \int_0^{2\pi} \int_0^R \varrho\, d\varrho\, d\phi = \int_0^{2\pi} \left[\tfrac{1}{2}\varrho^2\right]_0^R d\phi = \tfrac{1}{2} R^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \underline{\underline{\pi R^2}}$$

**Bsp.: 2.15** $\bar{\sigma} = \frac{Q}{A}$ mit $A = \tfrac{1}{2} \cdot a \cdot a = \frac{a^2}{2}$ ← mittlere Flächenladungsdichte

$$Q = \iint_A \sigma\, da = \int \int_{-x/2}^{x/2} \frac{2}{3} \sigma_{max} \left(1 + \frac{xy}{a^2}\right) dy\, dx = \frac{2}{3} \sigma_{max} \int_0^a \left[y + \frac{x}{a^2} \frac{y^2}{2}\right]_{-x/2}^{+x/2} dx$$

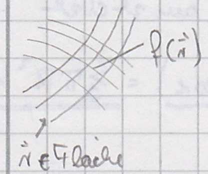
$$= \frac{2\sigma_{max}}{3a^2} \int_0^a \left[\frac{x}{2} + \cancel{\frac{x^3}{8a^2}} - \left(-\frac{x}{2} + \cancel{\frac{x^3}{8a^2}}\right)\right] dx = \frac{2}{3} \sigma_{max} \int_0^a x\, dx = \frac{2}{3} \sigma_{max} \underbrace{\tfrac{1}{2} a^2}_{= A} \Rightarrow$$

$$\Rightarrow \bar{\sigma} = \frac{Q}{A} = \underline{\underline{\frac{2}{3} \sigma_{max}}}$$

01.04.2014 W11 $\boxed{\text{grad} f \cdot \vec{ds} = df}$ $\xrightarrow{\vec{ds} = \hat{e}\, ds}$ $\boxed{\frac{df}{ds} = \hat{e} \cdot \text{grad} f}$ Richtungsableitung

$\int_C \vec{g}\, \vec{ds}$ $\hat{=}$ Linienintegral (C ← Raumkurve) => $\oint_{\partial A} \vec{g}\, \vec{ds}$ $\hat{=}$ Zirkulation des Vektorfelds ($\partial A$ ← Rand von A) für $C = \partial A$ $\hat{=}$ geschlossene Kurve

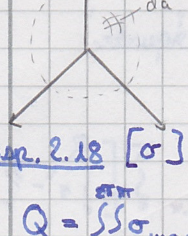
**Integration einer skalaren Funktion über gekrümmte Fläche**




$\boxed{\iint_A f(\vec{r})\, da}$ ← auf Zylinder Mantelfläche $da = \varrho\, d\phi\, dz$

$da = r^2 \sin\vartheta\, d\vartheta\, d\phi$ falls auf Kugeloberfläche

S44 **Bsp. 2.17**




$$A = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\vartheta\, d\vartheta\, d\phi = R^2\, 2\pi \left[-\cos\vartheta\right]_0^{\pi} = 2\pi R^2 (-(-1) + 1) = \underline{\underline{2\pi R^2}}$$

($r = R^2$)

**Bsp. 2.18** $[\sigma] = \frac{As}{m^2} = \frac{C}{m^2}$ => $Q = \sum \sigma \Delta a \to \iint \sigma\, da$

$$Q = \int_0^{2\pi} \int_0 \sigma_{max} \sin\vartheta\, \frac{1 + \cos^2(\phi/2)}{2}\, \underbrace{R^2 \sin\vartheta\, d\vartheta\, d\phi}_{da}$$